Antibandwidth Problem for Itchy Caterpillars

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***Abstract*—The antibandwidth problem is to label the vertices of a graph of n vertices by $1, 2, 3, \cdots, n$ bijectively, such that the minimum difference of labels of adjacent vertices is maximized. The antibandwidth problem is known as NP-hard for general graphs. In this paper, we give an antibandwidth labeling scheme for a special class of trees called itchy caterpillar and study the lower bound of antibandwidth problem for the same class of graphs. We also give exact results for some of its subclasses. The result for itchy caterpillars with hair length 1, is the first non-trivial exact result of antibandwidth problem for any class of graphs.**

I. Introduction

The antibandwidth problem [1] is a popular vertex labeling problem, where we have to label the vertices in such way that the minimum diffrence between two vertices is maximized. More formally, let G be a graph on n vertices. Given a bijection $f : V(G) \rightarrow \{1, 2, 3, \ldots, n\}$, if $|f| = \min\{|f(u) - f(v)| : uv \epsilon E(G)\}$ then the antibandwidth of G is the maximum $\{|f|\}$ over all such bijections f of G. In the literature, the antibandwidth problem is also known as dual bandwidth problem [2], separation number of graphs [3], maximum differential graph coloring problem [4].

The antibandwidth problem is NP-hard [3] for general graphs. It is known to be polynomially solvable for the complements of interval, arborescent comparability and threshold graphs [5], [6]. Exact results and tight bounds are known for paths, cycles, grid, meshes, hypercubes [1], [2], [4], [7]. For a complete binary tree with h height the antibandwidth is $2^h - 1$ [8] and for complete k-ary tree [9], the antibandwidth is $\frac{n+1-k}{2}$, when k is even. If $P_{m,n}$ is a tree, obtained from a path P_m and m copies of path P_n such that the i^{th} vertex of P_m is adjacent to an end vertex of the i^{th} copy of P_n. Then the antibandwidth of $P_{m,n}$ is $\lfloor \frac{mn}{2} \rfloor$ [8]. But above claim cannot be extended for graphs, when every vertex of P_m has more than one copies of P_n.

Millar *et al.* [10] introduced an antibandwidth labeling scheme for forests, which is known as Millar-Pritikin labeling scheme. According to Millar-Pritikin labeling scheme, for a forest G, if the bipartition sets are X and Y, where $|X| \leq |Y|$. Then the antibandwidth of G is at least $|X|$. They also showed that for a balanced bipartite graph ($|X| = |Y|$) Millar-Pritikin labeling scheme produces optimal antibandwidth value. Interestingly, still there is no results for any class of graphs, for which the optimal value of antibandwidth is non-trivial (i.e., not $\lfloor \frac{n}{2} \rfloor$).

Finding the antibandwidth of a graph has several practical applications. For example, if the vertices of the graph G represent sensitive facilities or chemicals, then placing them too close together can be risky. Formally the problem is known as enemy facility location problem [9]. Given a map, we define the country graph $G = (V, E)$ to be the undirected graph where countries are nodes and two countries are connected by an edge if they share a nontrivial boundary. We then consider the problem of assigning colors to nodes of G so that the color distance between nodes that share an edge is maximized [4]. Given n transmitters and n frequencies, the frequency assignment problem is to find a bijective frequency assignment where the interferring trasnmitters have as different frequency as possible, where transmitters are the vertices of graph G and there is an edge between two interfering transmitters [9].

A tree T is an *itchy caterpillar* if T can be decomposed into vertex disjoint paths $P_0, P_1, P_2, \cdots, P_k$ such that (a) exactly one end of a path P_i, $1 \leq i \leq k$, is adjacent to a vertex on P_0, (b) each P_i, $1 \leq i \leq k$, has equal number of vertices and (c) the number of paths among P_i, $1 \leq i \leq k$, adjacent to each vertex on P_0 is equal. P_0 is called *spine* and P_i is called *hair*. The tree shown in Figure 1 is an itchy caterpillar. Note that an itchy caterpillar is catterpilar if each P_i, $1 \leq i \leq k$, contains exactly one vertex. Itchy caterpillar is also a subclass of caterpilars with hair length at most r [11], where r is number of vertices in P_i, $1 \leq i \leq k$. In this paper, we provide the exact result of itchy caterpillars with hair length 1 which is the first non-trivial optimal result for any class of graph. Then we provide an antibandwidth labeling for itchy caterpillars with hair length 2. Based on that we generalize the labeling scheme for the entire class.

The rest of the paper is organized as follows. In Section 2, we describe some definations and preliminaries related to

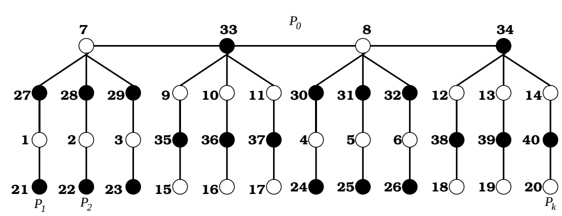

Fig. 1. An itchy caterpillar with antibandwidth labeling

the main result. In Section 3, we study the antibandwidth problem for itchy capterpillars with hair length 2 or more. In Section 4, we present the upper bound and exact result for itchy caterpillars with hair length 1. Finally Section 5 concludes the paper and discusses some open problems for future works.

II. Preliminaries

A *graph* G is a tuple (V, E) which consists of a finite set V of *vertices* and a finite set E of *edges*, each edge being an unordered pair of vertices.

A *tree* T is a connected graph which contains no cycle. A vertex u of T having degree one in T is called a *leaf* of T. A vertex u of T having degree greater than one in T is called an *internal vertex* of T. A *forest* is an acyclic graph. A tree is a connected forest, and every component of a forest is a tree.

A *caterpillar* is a tree for which deletion of leaves together with their incident edges produces a path. The *spine* of the caterpillar is the longest path to which all other vertices of the caterpillar are adjacent. The vertices which are on spine are called *spine vertices*. The vertices which are not on the spine are called *non-spine vertices* or *foot*. Every non-spine vertex is adjacent to exactly one spine vertex.

A tree T is an *itchy caterpillar* if T can be decomposed into vertex disjoint paths $P_0, P_1, P_2, \cdots, P_k$ such that (a) exactly one end of a path P_i, $1 \le i \le k$, is adjacent to a vertex on P_0, (b) each P_i, $1 \le i \le k$, has equal number of vertices and (c) the number of paths among P_i, $1 \le i \le k$, adjacent to each vertex on P_0 is equal. The tree shown in Figure 2 is an itchy caterpillar. P_0 is called the *spine* and P_i is called a *hair* of the itchy caterpillar. Let n be the total number of vertices, p be the number of spine vertices, q be the number of hairs attached to each spine vertex and r be the length of each hair of T. Then $n = pqr + p$ holds. A vertex v of an itchy caterpillar T resides in *level* L_0, when v is a spine vertex of T and v resides in *level* L_i, when v is in i^{th} position of a hair in the direction of spine to leaf, where $1 \le i \le r$.

Lemma 2.1. *Let G be an itchy caterpillar, p be the number of spine vertices, q be the number of hairs per spine vertices, r be the length of each hair and A, B be the two bipartion sets of G. For G, if A contains the vertices of Level L_{r-1} of hairs attached to odd spine vertices, then $|A| \le |B|$.*

Proof: Based on the values of p and r, we have the following three cases to consider:

Case I: *p is even.*

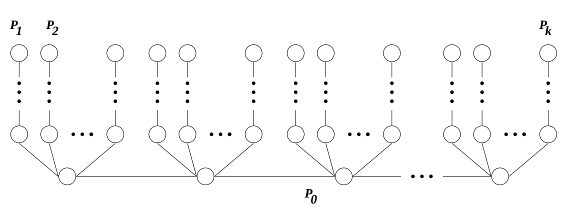

Fig. 2. An itchy caterpillar, composed with paths $P_0, P_1, \cdots, P_k$.

For p is even, we see that $|A| = |B|$, hence the claim is true.

Case II: *p is odd and r is even.*

When r is even, if A contains the vertices of Level L_{r-1} of hairs attached to odd spine vertices, then B contains the odd spine vertices. So the cardinality of A and B are $\left(\frac{r}{2}\right)pq + \left\lfloor\frac{p}{2}\right\rfloor$ and $\left(\frac{r}{2}\right)pq + \left\lceil\frac{p}{2}\right\rceil$ respectively. Clearly we see that, $|A| < |B|$.

Case III: *p is odd and r is odd.*

When r is odd, if A contains the vertices of Level L_{r-1}, of hairs attached to odd spine vertices, then A contains the odd spine vertices also. So the cardinality of A is $\left\lfloor\frac{r}{2}\right\rfloor * \left\lceil\frac{p}{2}\right\rceil * q + \left\lceil\frac{r}{2}\right\rceil * \left\lfloor\frac{p}{2}\right\rfloor * q + \left\lceil\frac{p}{2}\right\rceil$ and cardinality of B is $\left\lceil\frac{r}{2}\right\rceil * \left\lceil\frac{p}{2}\right\rceil * q + \left\lceil\frac{r}{2}\right\rceil * \left\lceil\frac{p}{2}\right\rceil * q + \left\lfloor\frac{p}{2}\right\rfloor$. Again we see that, $|A| < |B|$. Thus our proof is done. ■

III. Antibandwidth Labeling of Itchy Caterpillars

In this section, we provide an antibandwidth labeling scheme for general itchy caterpillars and study the lower bound of antibandwidth problem for the same class of graphs. First we define an antibandwidth labeling scheme for itchy caterpillars with hair length 2. Based on this scheme, we construct another labeling scheme for itchy caterpillars with hair length 2 or more. Finally We also show that for some cases, this labeling scheme produces optimal antibandwidth value.

Lemma 3.1. *Let G be an itchy caterpillar, p be the number of spine vertices, q be the number of hairs per spine vertices, r be the length of each hair, n be the total number of vertices of G. A and B be the two bipartition vertex set of G, such that $|A| \le |B|$. Then G admits a vertex labeling Γ with the following properties, when $r = 2$:*

1. *labels of the vertices of same set at Level L_i are consecutive from left to right.*
2. *labels of vertices of set A are from 1 to $|A|$ and labels for set B starts from $|A| + 1$ to n consecutively.*
3. *the lowest value of labels of vertices of A and the highest value of labels of vertices of B lie in Level L_{r-1}.*
4. *$|A| \le \delta_i \le |A| + \left\lceil\frac{p}{2}\right\rceil q$, where δ_i is the labeling difference between two neighbours of G in Γ.*

Proof: First we decompose the vertices of G into two bipartition vertex set A and B, such that the vertices of level L_1, of hairs attached to odd spine vertices, goes to A. Then according to Lemma 2.1, we see that $|A| \le |B|$.

Now we produce a vertex labeling Γ of the itchy caterpillar G, as shown in Figure 3, with the following labeling scheme.

Lebeling the vertices of set A:

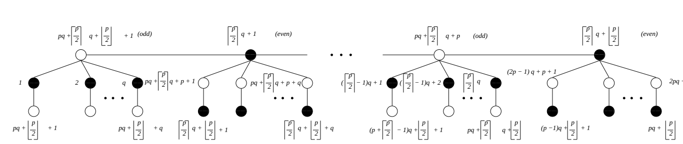

Fig. 3. Antibandwidth labeling of itchy caterpillar with hair length 2

- labels from 1 to $\left\lceil \frac{p}{2} \right\rceil * q$ are assigned to the vertices of Level L_1, of hairs attached to the odd spine vertices sequentially from left to right.
- labels from $\left\lceil \frac{p}{2} \right\rceil * q + 1$ are assigned to $\left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor$ to the even spine vertices sequentially from left to right.
- labels from $\left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + 1$ to $pq + \left\lfloor \frac{p}{2} \right\rfloor$ are assigned to the vertices of Level L_2, of hairs attached to the even spine vertices sequentially from left to right.

Lebeling the vertices of set B:

- labels from $pq + \left\lfloor \frac{p}{2} \right\rfloor + 1$ to $pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor$ are assigned to the vertices of Level L_2, of hairs attached to odd spine vertices sequentially from left to right.
- labels from $pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + 1$ to $pq + \left\lceil \frac{p}{2} \right\rceil * q + p$ are assigned to the odd spine vertices sequentially from left to right.
- labels from $pq + \left\lceil \frac{p}{2} \right\rceil * q + p + 1$ to $2pq + p$ are assigned to the vertices of Level L_2, of hairs attached to even spine vertices sequentially from left to right.

It is trivial to show that the above labeling scheme holds the propeties from 1 to 3. To prove property 4, we divide the edges of G into three sets E_1, E_2, E_3 as described below.

- E_1 contains the edges, having both end points in Level L_0.
- E_2 contains the edges, having one end point in Level L_0 and other in Level L_1.
- E_3 contains the edges, having one end point in Level L_1 and other in Level L_2.

Now based on the sets of edges of G, we have the following three cases:

Case I (For edges of E_1): We arrange the vertices of Level L_0 from left to right sequentially. According to the labeling scheme, the label of i^{th} vertex of Level L_0 is $\left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{i}{2} \right\rfloor$, when i is even and $pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{i}{2} \right\rceil$ when i is odd $(1 \le i \le p)$.

Now we decompose E_1 into E_{11} and E_{12}. So that, E_{11} contains the edges between $(i-1)^{th}$ and i^{th} vertices of Level L_0 and E_{12} contains the edges between i^{th} and $(i+1)^{th}$ vertices of Level L_0, where i is even and $1 < i \le p$.

For all the edges of E_{11} the labeling difference is

$$\begin{aligned} &= (pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{i-1}{2} \right\rceil) - (\left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{i}{2} \right\rfloor) \\ &= pq + \left\lfloor \frac{p}{2} \right\rfloor \end{aligned}$$

For all the edges of E_{12} the labeling difference is

$$\begin{aligned} &= (pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{i+1}{2} \right\rceil) - (\left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{i}{2} \right\rfloor) \\ &= pq + \left\lfloor \frac{p}{2} \right\rfloor + 1 \end{aligned}$$

As it can be trivially said that for G, $|A| = \left\lfloor \frac{n}{2} \right\rfloor$. We see that, $|A| \le (pq + \left\lfloor \frac{p}{2} \right\rfloor), (pq + \left\lfloor \frac{p}{2} \right\rfloor + 1) \le |A| + \left\lceil \frac{p}{2} \right\rceil * q$. Which concludes that, the claim is true for this case.

Case II (For edges of E_2): We arrange the even spine vertices sequentially from left to right, so that the label of i^{th} even spine vertex is $\left\lceil \frac{p}{2} \right\rceil * q + i$, where $1 \le i \le \left\lfloor \frac{p}{2} \right\rfloor$. We also arrange the vertices of Level L_1, attached with the even spine vertices sequentially from left to right, so that the label of j^{th} vertex of Level L_1, attached with i^{th} even spine vertex is $pq + \left\lceil \frac{p}{2} \right\rceil * q + p + (i-1)q + j$, where $1 \le j \le q$. So the labeling difference between the i^{th} even spine vertex and j^{th} vertex of Level L_1, attached with i^{th} even spine vertex is:

$$\begin{aligned} &= (pq + \left\lceil \frac{p}{2} \right\rceil * q + p + (i-1)q + j) - (\left\lceil \frac{p}{2} \right\rceil * q + i) \\ &= pq + p - q + i(q-1) + j \end{aligned}$$

Considering all the values of i and j, we see that $|A| \le pq + p - q + i(q-1) + j \le |A| + \left\lceil \frac{p}{2} \right\rceil * q$.

Similarly the labeling difference between the i^{th} odd spine vertex and j^{th} vertex of Level L_1, attached with i^{th} odd spine vertex is (where $1 \le i \le \left\lceil \frac{p}{2} \right\rceil$ and $1 \le j \le q$) :

$$\begin{aligned} &= (pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + i) - ((i-1)q + j) \\ &= pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + q + i(1-q) - j \end{aligned}$$

The minimum value of $pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor + q + i(1-q) - j$ is $pq + p$ when $i = \left\lceil \frac{p}{2} \right\rceil$, $j = q$ and the maximum value is $pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor$ when $i = 1$, $j = 1$. Again we see that $|A| \le (pq + p), (pq + \left\lceil \frac{p}{2} \right\rceil * q + \left\lfloor \frac{p}{2} \right\rfloor) \le |A| + \left\lceil \frac{p}{2} \right\rceil * q$.

Case III (For edges of E_3): In similar way as *case* 2, it can be proved that the claim is also true for all the edges of E_3. Thus concludes the proof. ■

Using Lemma 3.1, here we provide an antibandwidth labeling scheme for general itchy caterpillars with hair length 2 or more.

Theorem 3.2. *Let G be an itchy caterpillar, p be the number of spine vertices, q be the number of hairs per spine vertices, r be the length of each hair, n be the total number of vertices of G. A and B be the two bipartition vertex set of G, such that $|A| \le |B|$. Then G admits a vertex labeling Γ with the following properties, when $r \ge 2$:*

1. *labels of the vertices of same set at Level L_i are consecutive from left to right.*
2. *labels of vertices of set A are from 1 to $|A|$ and labels for set B starts from $|A| + 1$ to n consecutively.*
3. *the lowest value of labels of vertices of A and the highest value of labels of vertices of B lie in Level L_{r-1}.*
4. *$|A| \le \delta_i \le |A| + \left\lceil \frac{p}{2} \right\rceil q$, where δ_i is the labeling difference between two neighbours of G in Γ.*

Proof: First we decompose the vertices of G into two bipartition vertex set A and B, such that the vertices of level L_{r-1}, of hairs attached to odd spine vertices, goes to A. Then according to Lemma 2.1, we see that $|A| \le |B|$. Now we prove the claim by taking induction on r.

According to Lemma 3.1, We see that the claim is true, when $r = 2$.

Let the claim holds for an itchy caterpillar G', where the number of spine vertices is p, number of hairs attached to each spine vertex is q and the length of each hair is $r-1$. Then it is sufficient to prove that the claim also holds for an itchy caterpillar G, where the number of spine vertices is p, number of hairs attached to each spine vertex is q and the length of each hair is r.

Now using the labeling Γ' of G', here we produce an antibandwidth labeling Γ for G, as follows:

(1) construct G' from G by deleting all the leaves of G. Let A' and B' be the two bipartition vertex set of G'. Note that $|B| = |A'| + \left\lceil \frac{p}{2} \right\rceil q$ and $|A| = |B'| + \left\lfloor \frac{p}{2} \right\rfloor q$.
(2) as according to induction hypothesis, there is a labeling scheme Γ' for G' with the described properties of the claim, we label G' with Γ'.
(3) alter the labeling of vertices of G' such that, if the label of any vertex of set A' is i, then it becomes $i + |B'|$ and if the label of any vertex of set B' is j, then it becomes $j - |A'|$.
(4) shift the label of vertex set A' such that, if the label of any vertex of set A' is i, then it becomes $i + pq$.
(5) install the deleted vertices to produce G from G'.
(6) assign labels from $|B'|+1$ to $|B'| + \left\lfloor \frac{p}{2} \right\rfloor q$ to the leaves Set A at Level L_r sequentially from left to right and assign labels from $|B'| + \left\lfloor \frac{p}{2} \right\rfloor q + 1$ to $|B'| + pq$ to the leaves of Set B at Level L_r sequentially from left to right.

It can be trivially said that Γ holds the properties from 1 to 3.

Now we prove that Γ also holds the property 4. To do that, first we decompose the edges of G into two sets E_1, E_2 as follows:

- E_1 holds the edges of $E(G) \cap E(G')$. Note that $E(G) \cap E(G') = E(G')$.
- E_2 holds the edges other than E_1 (edges incident to the leaves of G).

Case I (for edges of E_1): here we prove that Γ holds the property 4, for all the edges of E_1.

Let an edge e_i is incident to vertex v_x and v_y, where $e_i \epsilon E_1$, $v_x \epsilon A'$, $v_y \epsilon B'$. In step 2, after producing the antibandwidth labeling Γ' of G', v_x is labeled with i and v_y is labeled with j, where $1 \le i \le |A'|$ and $|A'| + 1 \le j \le n$. Now according to the induction hypothesis, $|A'| \le j - i \le |A'| + \left\lceil \frac{p}{2} \right\rceil q$.

Now after doing steps 3 and 4, the label of vertex v_x becomes $i + |B'| + pq$ and the label of vertex v_y becomes $j - |A'|$. Now the labeling difference of two end points of edge e_i is:

$$i + |B'| + pq - (j - |A'|) = \quad |B'| + |A'| + pq - (j - i)$$

The value of $|B'| + |A'| + pq - (j - i)$ is maximized, when $(j - i) = |A'|$ and the highest value is:

$$\begin{aligned} &= |B'| + pq \\ &= |A| + \left\lceil \frac{p}{2} \right\rceil q, \text{as } |A| = |B'| + \left\lfloor \frac{p}{2} \right\rfloor q \end{aligned}$$

The lowest value of $|B'| + |A'| + pq - (j - i)$ is produced, when $(j - i) = |A'| + \left\lceil \frac{p}{2} \right\rceil q$ and that is:

$$\begin{aligned} &= |B'| + \left\lfloor \frac{p}{2} \right\rfloor q \\ &= |A|, \text{as } |A| = |B'| + \left\lfloor \frac{p}{2} \right\rfloor q \end{aligned}$$

This concludes that, Γ holds the property 4 for all the edges of E_1.

Case II (for edges of E_2): Here we prove that Γ, holds the property 4 for all the edges of E_2.

In step 5, we re-intsall the leaves of G. We see that the leaves of G residing in vertex Set B are adjacent to vertices of Set A of Level L_{r-1} and the leaves residing in vertex Set A are adjacent to vertices of Set B of Level L_{r-1}. In step 6, we label the leaves of Set A from $|B'| + 1$ to $|A|$. Property 1 and 3 of Γ ensures that the labels of their adjacent vertices are from $n - \left\lfloor \frac{p}{2} \right\rfloor q$ to n sequentially making the difference of $|B|$. Similarly the labeling difference of the leaves of Set B and their adjacent vertices is $|A|$. This concludes the proof. ∎

We observe that the lower bound produced by the labeling scheme described in Theorem 3.2 is the same as millar-pritikin labeling scheme. But the antibandwidth labeling scheme described in this paper is easy to understand and more natural for itchy caterpillars. The Corollary 3.3 shows that, for some cases the antibandwidth values, produced by these two labeling schemes are optimal. The proof of Corollary 3.3 is omitted in this paper, because of being trivial.

Corollary 3.3. *Let G be an itchy caterpillar, p be the number of spine vertices of G, q be the number of hairs per spine vertices of G, r be the length of hair of G, n be the number of vertices of G, then the antibandwidth of G is $\left\lfloor \frac{n}{2} \right\rfloor$ when,*

- *p is even*
- *p is odd, r is even.* ∎

IV. Itchy Caterpillars of Hair Length 1

In this section, we study the antibandwidth problem for itchy caterpillars of hair length 1. We provide an antibandwidth labeling scheme that produces optimal result for this class of graphs, which is the first non-trivial optimal result of antibandwidth problem for any graph class.

Theorem 4.1. *Let G be an itchy caterpillar, p be the number of spine vertices of G, q be the number of hairs per spine vertices of G, r be the length of hair of G, n be the number of vertices of G, then the antibandwidth of G is $\frac{n}{2}$ for p is even and $\left\lceil \frac{n-q}{2} \right\rceil$ for p is odd, when $r = 1$.*

We need the following lemmas to prove the above theorem.

Lemma 4.2. *Let G be an itchy caterpillar, p be the number of spine vertices of G, q be the number of hairs per spine* *vertices of G, r be the length of hair of G, n be the number of vertices of G. Then G admits a vertex labeling Γ, such that the difference of the labelings of any two adjacent vertices is at most $\left\lceil \frac{(n-q)}{2} \right\rceil$, when $r = 1$ and p is odd.*

Proof: To prove this lemma by contradiction, it is sufficent to prove that for an itchy caterpillar G, the minimum labelling difference $\left\lceil \frac{(n-q)}{2} \right\rceil + 1$ can not be acquired. To label the vertices of G, we have a set of labels, $N = \{1, 2, 3, \cdots, n\}$ with cardinality n.

To label G, first we split N into two sets N_1 and N_2 such that one of the set (say, N_1) has $\left\lceil \frac{(n-q)}{2} \right\rceil + 1$ number of labels and the other (say, N_2) has $n - \left\lceil \frac{(n-q)}{2} \right\rceil - 1$ number of labels, as shown in Figure 4. In Figure 4, the lower consecutive half of N are chosen for N_1 and the upper consecutive half of N are considered to be in N_2. Let, b be the number of spine vertices whose labels are chosen from N_1 and $1 \leq b \leq p$. For adjacent vertices of the spine vertices labeled from N_1 we are bound to choose labels from set N_2. Then the labels of N_1 can be distributed as described below.

- b number of labels, are assigned to the corresponding number of spine vertices of G.
- rest of the labels are assigned to the leaves of the other $p - b$ spine vertices (spine vertices for those, the labels will be assigned from N_2) and the number of such leaves is $(p - b) * q$.

The labels of N_2 can be distributed as described below:

- $p - b$ number of labels are assigned to the corresponding number of remaining spine vertices of G.
- $b * q$ number of labels are assigned to the leaves of other b spine vertices (spine vertices for those, the labels are assigned from N_1).

If we choose the labels in the way mentioned above, then no two neighboring vertices are assigned with the labels from N_1. Now considering the values that b can have, which corresponds to the arbitrarily choosing the number of labels for spine vertices from any of the halves of N, the following two cases can arise.

Case I: $\left\lceil \frac{p}{2} \right\rceil \leq b$.

In this case, the number of labels for the spine vertices chosen from N_1 is at least $\left\lceil \frac{p}{2} \right\rceil$. Let $b = \left\lceil \frac{p}{2} \right\rceil + i$, such that $0 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor$ and the number of labels needed to be in N_2 is $|N_2|'$. We can easily verify that the value of $|N_2|'$ must be,

$$\begin{aligned} &= (p - b) + (b * q) \\ &= (p - \left\lceil \frac{p}{2} \right\rceil - i) + \left(\left\lceil \frac{p}{2} \right\rceil + i \right) * q \\ &= \frac{n + q - 3}{2} + i(q - 1) + 1.\text{as } p \text{ is odd.} \end{aligned}$$

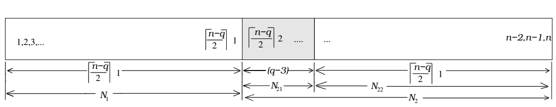

Fig. 4. Partitioning N into N_1 and N_2 (also N_2 into N_{21} and N_{22}).

But the value of $|N_2|$ is,

$$\begin{aligned} &= n - \left\lceil \frac{n-q}{2} \right\rceil - 1 \\ &= \frac{n + q - 3}{2}. \end{aligned}$$

Here we see $|N_2|' - |N_2| > 0$, which means the number of labels in N_2 is less than the number of labels needed to be in N_2 and thus contradicts the initial assumption.

Case II: $\left\lceil \frac{p}{2} \right\rceil > b$.

Similarly this case corresponds to the number of labels for the spine vertices chosen from N_1 is at most $\left\lfloor \frac{p}{2} \right\rfloor$. Let $b = i$, such that $0 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor$. The total number of labels needed to be in N_1 is $i + (p - i)q$. The minimum value of $i + (p - i)q$ is $\left\lfloor \frac{n+q}{2} \right\rfloor$ which can be found by putting, $i = \left\lfloor \frac{p}{2} \right\rfloor$. We see that this minimum value is greater than the cardinality of N_1. On the other hand, the total number of labels needed to be in N_2 is $(p - i) + iq$. The maximum value of which is $\left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor q$. We see that, this maximum value is less than the cardinality of N_2. At this point, as N_2 has extra labels, we divide N_2 into N_{21} and N_{22} as shown in Figure 4 and the labels from N_{21} are added with N_1 to meet up the shortage of N_1. From Figure 4, we see that this redistribution is possible without violating the initial assumption.

Let the number of labels needed to be in N_{21} is $|N_{21}|'$. We can easily verify that the minimum value (when $i = \left\lfloor \frac{p}{2} \right\rfloor$) of $|N_{21}|'$ must be,

$$\begin{aligned} &= b + (p - b) * q - \left\lceil \frac{(n-q)}{2} \right\rceil - 1 \\ &= pq + i(1 - q) - \left\lceil \frac{(n-q)}{2} \right\rceil - 1 \\ &= q - 2. \end{aligned}$$

But the value of $|N_{21}|$ is,

$$\begin{aligned} &= n - 2\left(\left\lceil \frac{n-q}{2} \right\rceil - 1 \right) \\ &= q - 3. \end{aligned}$$

We see $|N_{21}|' - |N_{21}| > 0$, which means the number of labels in N_{21} is less than the number of labels needed to be in N_{21}. Hence the initial assumption is violated and our proof is done. ■

Lemma 4.3. *Let G be an itchy caterpillar, p be the number of spine vertices of G, q be the number of hairs per spine vertices of G, r be the length of hair of G, n be the number of vertices of G. Then G admits a vertex labeling Γ, such that the difference between two neighbour vertices is at least $\frac{n}{2}$ for p is even and $\left\lceil \frac{n-q}{2} \right\rceil$ for p is odd, when $r = 1$.*

Proof: We produce a vertex labeling Γ of the itchy caterpillar G as shown in Figure 5, with the following labeling scheme.

- assign label 1, to the first even spine vertex of G. Continue labeling of the even spine vertices sequentially, so that last even spine vertex is labeled with $\left\lfloor \frac{p}{2} \right\rfloor$.

- label hair vertices of the odd spine vertices sequentially from $\lfloor \frac{p}{2} \rfloor + 1$ to $\lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil * q$.
- label hair vertices of the even spine vertices sequentially from $\lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil * q + 1$ to $\lfloor \frac{p}{2} \rfloor + p * q$.
- finally label the odd spine vertices sequentially from $\lfloor \frac{p}{2} \rfloor + p * q + 1$ to $p * q + p$.

Showing that, Γ is the vertex labeling of G, such that the difference between two neighbour vertices is at least $\frac{n}{2}$ for p is even and $\lceil \frac{n-q}{2} \rceil$ for p is odd in Γ, completes the proof of this lemma.

To do that, we divide all the edged $E(G)$ of G in vertex labeling Γ, into three sets E_1, E_2, E_3 as described below.

- E_1 holds the edges between even spine vertices and the vertices of their hairs.
- E_2 holds the edges between odd spine vertices and the vertices of their hairs.
- E_3 holds the edges bewteen odd and even spine vertices.

First we prove that for all the edges in E_1 the above claim is true. Let i and j are two numbers such that $1 \leq i \leq \lfloor \frac{p}{2} \rfloor$ and $1 \leq j \leq q$, where i is the i^{th} even vertex of the spine and j is the index of the vetex of j^{th} hair of that spine vertex, provided that $(i, j) \in E(G)$. So the label of vertex i is i and the label of vertex j is $\lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil * q + (i - 1) * q + j$ (see the itchy caterpillar of Figure 5). The difference between the labels of these two vertices is

$$
\begin{aligned}
&= \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{p}{2} \right\rceil * q + (i-1) * q + j - i \\
&= \left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{p}{2} \right\rceil * q + j + (i-1) * q - i
\end{aligned}
$$

For $i = 1$ and $j = 1$, we get the minimum value of the above expression that is $\lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil * q$. Which states that the minimum value of the differences between the vertices of end points of all the edges in E_1 is $\lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil * q$. We can easily verify that $\lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil * q \geq \frac{n}{2}$ for any value of p and q. Simillarly, we can easily show that, all the edges (which represent labeling difference of their end point vertices) in E_2 and E_3 also satisfies the claim. ■

Proof of Theorem 4.1. For an itchy caterpillar, when p is even and $r = 1$, Lemma 4.3 says that the antibandwidth is at least $\frac{n}{2}$, which is the trivial upper bound of antibandwidth for any class of graphs. Considering this with Lemma 4.2 (non-trivial upper bound for itchy caterpillars, when p is odd and $r = 1$) along with Lemma 4.3 concludes the proof of Theorem 4.1.

We have the following corollary from Theorem 4.1, which is trivial to show.

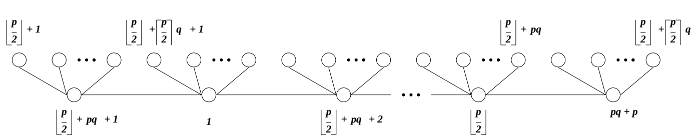

Fig. 5. An itchy caterpillar with hair length 1

Corollary 4.4. *Millar-Pritikin labeling scheme produces exact antibandwidth value for itchy caterpillar with hair length* 1. ■

V. Conclusion

In this paper, first we provided labeling schemes for itchy caterpillars with hair length 2, and then we generalized the labeling scheme for itchy caterpillars with hair length 2 or more. We derived the lower bound for antibandwidth problem, using the defined labeling scheme and showed that, for some cases the result is optimal. Then we showed that the optimal value of antibandwidth for itchy caterpillars with hair length 1, is $\lfloor \frac{n}{2} \rfloor$ when p is even and $\left\lceil \frac{(n-q)}{2} \right\rceil$ when p is odd. $\left\lceil \frac{(n-q)}{2} \right\rceil$ is the first non-trivial upper bound of antibandwidth problem for any class of graphs. We also showed that Millar and Pritikin labeling scheme described in [10] also produces same set of results. But the labeling scheme provided in this paper are more intuitive, easy to understand and natural for the defined class of graphs.

The antibandwidth problem still offers a wide range of interesting open problems. It will be very interesting to find any other graph classes, for which the antibandwidth is non-trivial. Finding any general upper bound for the entire class of itchy caterpillars is also open. Beside these, the hardness for finding antibandwidth of general caterpillars, is also interesting to investigate.

References

[1] A. Raspaud, H. Schroder, O. Sÿkora, L. Török, and I. Vrťo, “Antibandwidth and cyclic antibandwidth of meshes and hypercubes,” *Discrete Mathematics.*, vol. 309, no. 11, pp. 3541–3552, 2009.

[2] L. Yixun and Y. Jinjiang, “The dual bandwidth problem for graphs,” *Journal of Zhengzhou University (Natural Science Edition)*, vol. 35, pp. 1–5, 2003.

[3] J. Y.-T. Leung, O. Vornberger, and J. Witthoff, “On some variants of the bandwidth minimization problem,” *SIAM Journal on Computing*, vol. 13, no. 3, pp. 650–667, 1984.

[4] Y. Hun, S. Kobourov, and S. Veeramon, “On maximum differential graph coloring,” *GD’10 Proceedings of the 18th international conference on Graph drawing*, pp. 274–286, 2010.

[5] S. Donnelly and G. Isaak, “Hamiltonian powers in threshold and arborescent comparability graphs,” *Discrete Mathematics*, vol. 202, no. 1-3, pp. 33–44, 1999.

[6] G. Isaak, “Powers of hamiltonian paths in interval graphs,” *Journal of Graph Theory*, vol. 28, no. 1, pp. 31–38, 1998.

[7] L. Palios, “Upper and lower bounds for the optimal tree partitions,” *Technical Report GCG*, p. 46, 1994.

[8] Y. Weili, Z. Ju, and L. Xiaoxu, “Dual bandwith of some special trees,” *Journal of Zhengzhou University (Natural Science Edition)*, vol. 35, 2003.

[9] T. Calamoneri, A. Massini, L. Török, and I. Vrťo, “Antibandwidth of complete k-ary trees,” *Discrete Mathematics*, vol. 309, no. 22, pp. 6408–6414, 2009.

[10] Z. Miller and D. Pritikin, “On separation number of a graph,” *Networks*, vol. 19, no. 6, pp. 651–666, 1989.

[11] B. Monie, “The bandwith minimization problem for caterpillars with hair length 3 is np-complete,” *SIAM Journal on Algebraic and Discrete Methods*, vol. 7, no. 4, pp. 505–512, 1986.